Ex-2 P.T $3x^2 + 7y^2 + 3z^2 + 10yz - 2zx + 10xy + 4x - 12y - 4z + 1 = 0$ to the standard form and state the nature of the surface represented by it.

→ The given equation is:

$$3x^2 + 7y^2 + 3z^2 + 10yz - 2zx + 10xy + 4x - 12y - 4z + 1 = 0.$$

comparing it with

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

$a = 3,\ b = 7,\ c = 3,\ f = 5,\ g = -1$

$h = 5,\ u = 2,\ v = -6,\ w = -2,\ d = 1$

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 3 & 5 & -1 \\ 5 & 7 & 5 \\ -1 & 5 & 3 \end{vmatrix}$$

$$= 3\begin{vmatrix} 7 & 5 \\ 5 & 3 \end{vmatrix} - 5\begin{vmatrix} 5 & 5 \\ -1 & 3 \end{vmatrix} - 1\begin{vmatrix} 5 & 7 \\ -1 & 5 \end{vmatrix}$$

$$= 3[21 - 25] - 5[15 + 5] - 1[25 + 7]$$

$$= 3(-4) - 5(20) - 1(32)$$

$$= -12 - 100 - 32$$

$$= -12 - 132 = -144 \neq 0$$

The discriminating cubic is

$$\begin{vmatrix} 3-\lambda & 5 & -1 \\ 5 & 7-\lambda & 5 \\ -1 & 5 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\begin{vmatrix} 7-\lambda & 5 \\ 5 & 3-\lambda \end{vmatrix} - 5\begin{vmatrix} 5 & 5 \\ -1 & 3-\lambda \end{vmatrix} - 1\begin{vmatrix} 5 & 7-\lambda \\ -1 & 5 \end{vmatrix} = 0$$

$$(3-\lambda)\left[(7-\lambda)(3-\lambda) - 25\right] - 5\left[5(3-\lambda) + 5\right] - 1\left[25 + (7-\lambda)\right] = 0$$

$$(3-\lambda)\left[21 - 7\lambda - 3\lambda + \lambda^2 - 25\right] - 5\left[15 - 5\lambda + 5\right] - 1\left[25 + 7 - \lambda\right] = 0$$

$$(3-\lambda)\left[\lambda^2 - 10\lambda - 4\right] - 5\left[-5\lambda + 20\right] - 1\left[-\lambda + 32\right] = 0$$

$$3\lambda^2 - 30\lambda - 12 - \lambda^3 + 10\lambda^2 + 4\lambda + 25\lambda - 100 + \lambda - 32 = 0$$

$$-\lambda^3 + 13\lambda^2 - 26\lambda + 26\lambda - 144 = 0$$

$$\lambda^3 - 13\lambda^2 + 144 = 0$$

$$(\lambda+3)(\lambda-4)(\lambda-12) = 0$$

$$\lambda = -3, 4, 12$$

The central planes are

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = 0$$

$$6x + 10y - 2z + 4 = 0$$

$$\Rightarrow \quad 3x + 5y - z + 2 = 0 \quad \text{--- (1)}$$

$$14y + 10z + 10x - 12 = 0$$

$$\Rightarrow \quad 5x + 7y + 5z - 6 = 0 \quad \text{--- (2)}$$

$$6z + 10y - 2x - 4 = 0$$

$$\Rightarrow \quad x - 5y - 3z + 2 = 0 \quad \text{--- (3)}$$

Multiply (1) by 3 and (3) by 1, we get

$$9x + 15y - 3z + 6 = 0 \quad \text{--- (4)}$$

$$x - 5y - 3z + 2 = 0 \quad \text{--- (5)}$$

Subtracting (5) from (4), we get

$$8x + 20y + 4 = 0$$

$$2x + 5y + 1 = 0 \quad \text{--- (6)}$$

Multiplying (1) by 5 and (3) by 1, we get

$$15x + 25y - 5z + 18 = 0 \quad \text{--- (7)}$$

$$5x + 7y + 5z - 6 = 0 \quad \text{--- (8)}$$

Adding (7) and (8) we get

$$20x + 32y + 4 = 0$$

$$5x + 8y + 1 = 0 \quad \text{--- (9)}$$

Solving (6) and (9), we get

$$\frac{x}{5-8} = \frac{y}{5-2} = \frac{1}{16-25}$$

$$\frac{x}{-3} = \frac{y}{3} = \frac{1}{-9}$$

$$\frac{x}{1} = \frac{y}{-1} = \frac{1}{3}$$

$$x = \frac{1}{3}, \quad y = \frac{-1}{3}$$

from (1) $\quad 1 - \frac{8}{3} - z + 2 = 0$

$$z = \frac{4}{3}$$

$$x = \frac{1}{3}, \quad y = \frac{-1}{3}, \quad z = \frac{4}{3}$$

centre is $\left(\frac{1}{3}, \frac{-1}{3}, \frac{4}{3}\right)$

$$\alpha = \frac{1}{3}, \quad \beta = \frac{-1}{3}, \quad \gamma = \frac{4}{3}$$

Now, $d' = u\alpha + v\beta + w\gamma + d$

$$= (2)\left(\frac{1}{3}\right) + (-6)\left(\frac{-1}{3}\right) + (-2)\left(\frac{4}{3}\right) + 1$$

$$= \frac{2}{3} + 2 - \frac{8}{3} + 1$$

$$= \frac{8}{3} - \frac{8}{3} + 1 = 1$$

Take $\lambda_1 = -3$, $\lambda_2 = 4$, $\lambda_3 = 12$

On suitable rotation of the axis, the eqn is reduced to

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0$$

$$-3x^2 + 4y^2 + 12z^2 + 1 = 0$$

$$3x^2 - 4y^2 - 12z^2 = 1$$

which is hyperboloid of two sheets.

Ex·3 Reduce the equation

$$2x^2 - 7y^2 + 2z^2 - 10yz - 8zx - 10xy + 6x + 12y - 6z + 5 = 0$$

to the standard form and give the nature of the surface.

→ The given eqn is -

$$2x^2 - 7y^2 + 2z^2 - 10yz - 8zx - 10xy + 6x + 12y - 6z + 5 = 0$$

comparing it with

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$ we get,

$$\begin{vmatrix} 2-\lambda & -5 & -4 \\ -5 & -7-\lambda & -5 \\ -4 & -5 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\begin{vmatrix} -7-\lambda & -5 \\ -5 & 2-\lambda \end{vmatrix} + 5\begin{vmatrix} -5 & -5 \\ -4 & 2-\lambda \end{vmatrix} - 4\begin{vmatrix} -5 & -7-\lambda \\ -4 & -5 \end{vmatrix} = 0$$

$$(2-\lambda)\left[(-7-\lambda)(2-\lambda) - 25\right] + 5\left[-5(2-\lambda) - 20\right] - 4\left[25 - (-4)(-7-\lambda)\right] = 0$$

$$(2-\lambda)\left[-14+7\lambda-2\lambda+\lambda^2-25\right] + 5\left[-10+5\lambda-20\right] - 4\left[25-(28+4\lambda)\right] = 0$$

$$\lambda^3 + 3\lambda^2 - 90\lambda + 216 = 0$$

$$(\lambda-3)(\lambda-6)(\lambda+12) = 0$$

$$\lambda = 3, 6, -12$$

The Centre is obtained from eqn.

$$\frac{\partial f}{\partial x} = 0 \quad \frac{\partial f}{\partial y} = 0 \quad \frac{\partial f}{\partial z} = 0$$

$$2x - 5y - 4z + 3 = 0 \quad \text{——} \ (1)$$
$$5x + 7y + 5z - 6 = 0 \quad \text{——} \ (2)$$
$$4x + 5y - 2z + 3 = 0 \quad \text{——} \ (3)$$

Multiplying (1) by 5 and (2) by 4, we get

$$10x - 25y - 20z + 15 = 0 \quad \text{——} \ (4)$$
$$20x + 28y + 20z - 24 = 0 \quad \text{——} \ (5)$$

$a = 2, \; b = -7, \; c = 2, \; f = -5, \; g = -4$

$h = -5, \; u = 3, \; v = 6, \; w = -3, \; d = 5$

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -5 & -4 \\ -5 & -7 & -5 \\ -4 & -5 & 2 \end{vmatrix}$$

$$= 2 \begin{vmatrix} -7 & -5 \\ -5 & 2 \end{vmatrix} + 5 \begin{vmatrix} -5 & -5 \\ -4 & 2 \end{vmatrix} - 4 \begin{vmatrix} -5 & -7 \\ -4 & -5 \end{vmatrix}$$

$$= 2[-14 - 25] + 5[-10 - 20] - 4[25 - 28]$$

$$= 2[-39] + 5(-30) - 4(-3)$$

$$= -78 - 150 + 12$$

$$= -216 \neq 0$$

The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$$

Adding (4) and (5)

$$30x + 3y - 9 = 0$$

$$10x + y - 3 = 0 \quad \text{— (6)}$$

Multiplying (5) by 2 and (3) by 5, we get

$$10x + 14y + 10z - 12 = 0 \quad \text{— (7)}$$

$$20x + 25y - 10z + 15 = 0 \quad \text{— (8)}$$

Adding (7) and (8)

$$30x + 39y + 3 = 0$$

$$10x + 13 + 1 = 0 \quad \text{— (9)}$$

Subtracting (6) from (9), we get

$$12y + 4 = 0$$

$$y = -\frac{1}{3}$$

from (6) $10x - \frac{1}{3} - 3 = 0$

$$10x = \frac{10}{3}$$

$$x = \frac{1}{3}$$

from (3), $\frac{4}{3} - \frac{5}{3} - 2z + 3 = 0$

$$2z = \frac{8}{3} \Rightarrow z = \frac{4}{3}$$

Centre is $\left(\frac{1}{3}, \frac{-1}{3}, \frac{4}{3}\right)$

$\alpha = \frac{1}{3}$, $\beta = \frac{-1}{3}$, $\gamma = \frac{4}{3}$

$$d' = u\alpha + v\beta + w\gamma + d$$

$$= (3)\left(\frac{1}{3}\right) + 6\left(\frac{-1}{3}\right) + (-3)\left(\frac{4}{3}\right) + 5$$

$$= 1 - 2 - 4 + 5 = 0$$

On suitable rotation of axes, the given equation is transformed to

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0$$

$$3x^2 + 6y^2 - 12z^2 = 0$$

$$x^2 + 2y^2 - 4z^2 = 0$$

which is a cone with vertex $\left(\frac{1}{3}, \frac{-1}{3}, \frac{4}{3}\right)$